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Periodic Boundary Value Problems for Second Order Impulsive Integrodifferential Equations of Mixed Type in Banach Spaces

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In this paper, the author obtains an existence theorem of minimal and maximal solutions for the periodic boundary value problems of second order impulsive integrodifferential equations of mixed type in Banach space by means of the monotone iterative technique and cone theory based on a comparison result.

1. INTRODUCTION

The theory of impulsive differential equations is an important branch of differential equations (see [1]). In this paper, we consider the existence of minimal and maximal solutions for the periodic boundary value problems (PBVP) of second order impulsive integrodifferential equations of mixed type in Banach space

$$\begin{aligned} &-u'' = f(t, u, Tu, Su), \qquad t \neq t_k, \\ &\Delta u|_{t=t_k} = I_k(u(t_k)), \\ &\Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), \qquad k = 1, 2, \ldots, m, \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned} \tag{1}$$

where $f \in C[J \times E \times E \times E, E]$, $J = [0, 2\pi]$, E is a real Banach space, $0 < t_1 < \ldots < t_m < 2\pi$, $I_k \in C[E, E]$, $\bar{I}_k \in C[E, E]$, $\Delta u|_{t=t_k} = u(t_k^+) -$

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$u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $k = 1, 2, \ldots, m$, and the operators T, S are given by

$$Tu(t) = \int_0^t k(t, s)u(s)\, ds, \qquad Su(t) = \int_0^{2\pi} k_1(t, s)u(s)\, ds, \tag{2}$$

with $k \in C[D, R]$, $D = \{(t, s) \in R^2 \mid 0 \leq s \leq t \leq 2\pi\}$, $k_1 \in C[J \times J, R]$. R denotes the real numbers. In the special case where $E = R$, $I_k = \bar{I}_k = 0$, $k = 1, 2, \ldots, m$, the minimal and maximal solutions of PBVP (1) have been obtained by means of the monotone iterative technique based on a comparison result in [2]. But, it is easy to see that the comparison result in [2] is not applicable in the impulsive case. Therefore, in this paper, we shall obtain a comparison result for the impulsive case, and then, we obtain an existence theorem of minimal and maximal solutions of PBVP (1) by means of the monotone iterative technique and cone theory. Finally, we give an example for applying this existence theorem.

2. Several Lemmas

Let $PC^1[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuously differentiable at $t \neq t_k$ and left continuous at $t = t_k$, and $u(t_k^+), u'(t_k^-), u'(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$. Evidently, $PC^1[J, E]$ is a Banach space with norm

$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\},$$

where

$$\|u\|_{pc} = \sup_{t \in J} \|u(t)\|, \qquad \|u'\|_{pc} = \sup_{t \in J} \|u'(t)\|.$$

Note that $PC[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$ is also a Banach space with norm

$$\|u\|_{pc} = \sup_{t \in J} \|u(t)\|.$$

Let the Banach space E be partially ordered by a cone P of E, i.e., $u \leq v$ if and only if $v - u \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$. Let $K = \{u \in PC[J, E] : u(t) \geq \theta$ for all $t \in J)$. Then K is a cone in space $PC[J, E]$, and $PC[J, E]$ is partially ordered by K: $u \leq v$ iff $v - u \in K$; i.e., $u(t) \leq v(t)$ for all $t \in J$. Evidently, if

P is normal, then K is also normal. The properties of the cone and the partial order may be found in [3]. Let $J' = J\backslash\{t_1, t_2, \ldots, t_m\}$. A map $u \in PC^1[J, E] \cap C^2[J', E]$ is called a solution of PBVP (1) if it satisfies (1).

Consider the PBVP

$$\begin{aligned} &-u'' + M^2u = g(t), \qquad t \neq t_k, \\ &\Delta u|_{t=t_k} = I_k(u(t_k)), \\ &\Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), \qquad k = 1, 2, \ldots, m, \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned} \tag{3}$$

where $M > 0$ is constant, $g \in PC[J, E]$. For convenience, we denote $I_k = I_k(u(t_k)), \bar{I}_k = \bar{I}_k(u(t_k))$.

LEMMA 1. *$u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of* PBVP (3) *if and only if $u \in PC[J, E]$ is a solution of the impulsive integral equation*

$$u(t) = \int_0^{2\pi} G(t,s)g(s)\,ds + \sum_{k=1}^{m} [G(t,t_k)(-\bar{I}_k) + H(t,t_k)I_k], \tag{4}$$

where

$$G(t,s) = (2M(e^{2\pi M} - 1))^{-1} \begin{cases} e^{M(2\pi - t + s)} + e^{M(t-s)}, & 0 \le s < t \le 2\pi, \\ e^{M(2\pi + t - s)} + e^{M(s-t)}, & 0 \le t \le s < 2\pi, \end{cases} \tag{5}$$

$$H(t,s) = (2(e^{2\pi M} - 1))^{-1} \begin{cases} e^{M(2\pi - t + s)} - e^{M(t-s)}, & 0 \le s < t \le 2\pi, \\ e^{M(s-t)} - e^{M(2\pi + t - s)}, & 0 \le t \le s < 2\pi. \end{cases} \tag{6}$$

Proof. First, suppose that $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of PBVP (3). Then

$$-u'' + M^2u = g(t), \qquad t \neq t_k,$$

i.e.,

$$(e^{-2Mt}(e^{Mt}u(t))')' = -g(t)e^{-Mt}, \qquad t \neq t_k.$$

Let

$$y(t) = e^{-2Mt}(e^{Mt}u(t))' = e^{-Mt}(u'(t) + Mu(t)). \tag{7}$$

Then

$$y'(t) = -g(t)e^{-Mt}, \qquad t \neq t_k. \tag{8}$$

It is easy to see by integration of (8) that

$$y(t_1) - y(0) = -\int_0^{t_1} g(s)e^{-Ms}\,ds,$$

$$y(t) - y(t_1^+) = -\int_{t_1}^{t} g(s)e^{-Ms}\,ds, \qquad t_1 < t \leq t_2.$$

So

$$y(t) = y(0) - \int_0^t g(s)e^{-Ms}\,ds + y(t_1^+) - y(t_1), \qquad t_1 < t \leq t_2.$$

In the same way, we can show that

$$y(t) = y(0) - \int_0^t g(s)e^{-Ms}\,ds + \sum_{0<t_k<t} I_k^*, \qquad t \in J, \tag{9}$$

where $y(0) = u'(0) + Mu(0)$, $I_k^* = I_k^*(y(t_k)) = e^{-Mt_k}(\bar{I}_k + MI_k)$. In view of (7), we have

$$(e^{Mt}u(t))' = e^{2Mt}(y(0) - \int_0^t g(s)e^{-Ms}\,ds + \sum_{0<t_k<t} I_k^*).$$

Let

$$z(t) = e^{Mt}u(t), \qquad m(t) = e^{2Mt}(y(0) - \int_0^t g(s)e^{-Ms}\,ds + \sum_{0<t_k<t} I_k^*). \tag{10}$$

Then

$$\Delta z|_{t=t_k} = e^{Mt_k}I_k(u(t_k)) = I_k^{**}.$$

Similarly, we get the formula

$$z(t) = z(0) + \int_0^t m(s)\,ds + \sum_{0<t_k<t} I_k^{**}, \qquad t \in J,$$

so

$$u(t) = z(t)e^{-Mt}\Big(u(0) + \int_0^t m(s)\,ds + \sum_{0<t_k<t} e^{Mt_k} I_k\Big), \qquad t \in J. \tag{11}$$

By calculation, we can get

$$\begin{aligned}\int_0^t m(s)\,ds = (2M)^{-1}\Big\{y(0)(e^{2Mt} - 1) + \int_0^t e^{Ms} g(s)\,ds \\ - e^{2Mt}\int_0^t e^{-Ms} g(s)\,ds + \sum_{0<t_k<t} (e^{2Mt} - e^{2Mt_k}) I_k^*\Big\}.\end{aligned} \tag{12}$$

Substituting (12) into (11), we obtain

$$\begin{aligned} u(t) &= (2M)^{-1}\Big\{(Mu(0) - u'(0))e^{-Mt} + (Mu(0) + u'(0))e^{Mt} \\ &\quad + e^{-Mt}\int_0^t e^{Ms} g(s)\,ds - e^{Mt}\int_0^t e^{-Ms} g(s)\,ds \\ &\quad + \sum_{0<t_k<t} e^{M(t-t_k)}(\bar{I}_k + MI_k) - \sum_{0<t_k<t} e^{-M(t-t_k)}(\bar{I}_k - MI_k)\Big\}, \qquad t \in J, \\ u'(t) &= 2^{-1}\Big\{-(Mu(0) - u'(0))e^{-Mt} + (Mu(0) + u'(0))e^{Mt} \\ &\quad - e^{-Mt}\int_0^t e^{Ms} g(s)\,ds - e^{Mt}\int_0^t e^{-Ms} g(s)\,ds + \sum_{0<t_k<t} e^{M(t-t_k)}(\bar{I}_k + MI_k) \\ &\quad + \sum_{0<t_k<t} e^{-M(t-t_k)}(\bar{I}_k - MI_k)\Big\}, t \in J. \end{aligned} \tag{13}$$

In view of $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, we have

$$\begin{aligned} Mu(0) + u'(0) = e^{2\pi M}(e^{2\pi M} - 1)^{-1}\Big[\int_0^{2\pi} e^{-Ms} g(s)\,ds \\ - \sum_{0<t_k<2\pi} e^{-Mt_k}(\bar{I}_k + MI_k)\Big], \end{aligned} \tag{14}$$

$$Mu(0) - u'(0) = (e^{2\pi M} - 1)^{-1}\Big[\int_0^{2\pi} e^{Ms} g(s)\,ds - \sum_{0<t_k<2\pi} e^{Mt_k}(\bar{I}_k - MI_k)\Big]. \tag{15}$$

Substituting (14) and (15) into (13), and making use of the fact that

$$\sum_{k=1}^{m} I_k = \sum_{0<t_k<2\pi} I_k = \sum_{0<t_k<t} I_k + \sum_{t\le t_k<2\pi} I_k,$$

we obtain

$$\begin{aligned}
u(t) &= [2M(e^{2\pi M}-1)]^{-1}\Big\{\int_0^t e^{M(2\pi-t+s)}g(s)\,ds + \int_t^{2\pi} e^{M(s-t)}g(s)\,ds \\
&\quad + \int_0^t e^{M(t-s)}g(s)\,ds + \int_t^{2\pi} e^{M(2\pi+t-s)}g(s)\,ds \\
&\quad + \sum_{0<t_k<t} e^{M(2\pi-t+t_k)}(MI_k - \bar I_k) + \sum_{t\le t_k<2\pi} e^{-M(t-t_k)}(MI_k - \bar I_k) \\
&\quad - \sum_{0<t_k<t} e^{M(t-t_k)}(MI_k + \bar I_k) - \sum_{t\le t_k<2\pi} e^{M(2\pi+t-t_k)}(MI_k + \bar I_k)\Big\} \\
&= [2M(e^{2\pi M}-1)]^{-1}\Big\{\int_0^t [e^{M(2\pi-t+s)} + e^{M(t-s)}]g(s)\,ds \\
&\quad + \int_t^{2\pi} [e^{M(s-t)} + e^{M(2\pi+t-s)}]g(s)\,ds + \sum_{0<t_k<t} [e^{M(2\pi-t+t_k)} + e^{M(t-t_k)}](-\bar I_k) \\
&\quad + \sum_{t\le t_k<2\pi} [e^{-M(t-t_k)} + e^{M(2\pi+t-t_k)}](-\bar I_k) + M\sum_{0<t_k<t} [e^{M(2\pi-t+t_k)} - e^{M(t-t_k)}]I_k \\
&\quad + M\sum_{t\le t_k<2\pi} [e^{-M(t-t_k)} - e^{M(2\pi+t-t_k)}]I_k\Big\} \\
&= \int_0^{2\pi} G(t,s)g(s)\,ds + \sum_{k=1}^{m} [G(t,t_k)(-\bar I_k) + H(t,t_k)I_k],
\end{aligned}$$

i.e., $u(t)$ is a solution of Eq. (4).

Conversely, assume that $u \in PC[J, E]$ is a solution of Eq. (4). It is easy to see that

$$\Delta\left[\sum_{k=1}^{m} G(t,t_k)(-\bar I_k)\right]\Bigg|_{t=t_k} = \theta, \tag{16}$$

$$\Delta\left[\sum_{k=1}^{m} H(t,t_k)I_k\right]\Bigg|_{t=t_k} = I_k, \qquad k = 1,2,\ldots,m. \tag{17}$$

Direct differentiation on (4) implies, for $t \neq t_k$,

$$
\begin{aligned}
u'(t) &= \int_0^{2\pi} G_t(t,s)g(s)\,ds + \sum_{k=1}^{m} [G_t(t,t_k)(-\bar{I}_k) + H_t(t,t_k)I_k] \\
&= -\int_0^{2\pi} H(t,s)g(s)\,ds + \sum_{k=1}^{m} [H(t,t_k)(\bar{I}_k) - M^2G(t,t_k)I_k] \\
u''(t) &= -g(t) + M^2\int_0^{2\pi} G(t,s)g(s)\,ds + M^2\sum_{k=1}^{m} [G(t,t_k)(-\bar{I}_k) + H(t,t_k)I_k] \\
&= -g(t) + M^2u(t), \qquad t \neq t_k.
\end{aligned}
\tag{18}
$$

Noting (4), (16), (17), and (18), we have

$$
\Delta u|_{t=t_k} = I_k(u(t_k)), \qquad \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)).
$$

Making use of the fact $G(0, s) = G(2\pi, s)$, $H(0, s) = H(2\pi, s)$ for $s \in J$, and of (4) and (18), we obtain

$$
u(0) = u(2\pi), \qquad u'(0) = u'(2\pi).
$$

Hence, $u(t) \in PC^1[J, E] \cap C^2[J', E]$ is a solution of Eq. (3). ∎

Consider the linear PBVP

$$
\begin{aligned}
&-u'' + M^2u = -NTu - N_1Su + h(t), && t \neq t_k, \\
&\Delta u|_{t=t_k} = I_k(\eta(t_k)), && \\
&\Delta u'|_{t=t_k} = \bar{I}_k(\eta(t_k)), && k = 1, 2, \ldots, m, \\
&u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), &&
\end{aligned}
\tag{19}
$$

where $M > 0$, $N \geq 0$, $N_1 \geq 0$ are constant, $h, \eta \in PC[J, E]$.

In the following denote

$$
k^* = \max_{(t,s)\in D} |k(t,s)|, \qquad k_1^* = \max_{(t,s)\in J\times J} |k_1(t,s)|.
$$

LEMMA 2. *If*

$$
Nk^* + N_1k_1^* < M^2(2\pi)^{-1}, \tag{20}
$$

then the linear PBVP (19) *has exactly one solution* $u \in PC^1[J, E] \cap C^2[J', E]$ *given by*

$$u(t) = \int_0^{2\pi} G^*(t,s)h(s)\,ds + \sum_{k=1}^{m} [G^*(t,t_k)(-\bar{I}_k) + H^*(t,t_k)I_k], \tag{21}$$

where

$$G^*(t,s) = G(t,s) + F(t,s), \tag{22}$$

$$H^*(t,s) = H(t,s) + F_1(t,s), \tag{23}$$

$$F(t,s) = \int_0^{2\pi} Q(t,r)G(r,s)\,dr, \qquad F_1(t,s) = \int_0^{2\pi} Q(t,r)H(r,s)\,dr, \tag{24}$$

$$Q(t,s) = \sum_{n=1}^{\infty} k_2^{(n)}(t,s), \tag{25}$$

$$k_2^{(n)}(t,s) = \int_0^{2\pi} \cdots \int_0^{2\pi} k_2(t,r_1)k_2(r_1,r_2) \ldots k_2(r_{n-1},s)\,dr_1 \ldots dr_{n-1}, \tag{26}$$

$$k_2(t,s) = -N\int_s^{2\pi} G(t,r)k(r,s)\,dr - N_1\int_0^{2\pi} G(t,r)k_1(r,s)\,dr. \tag{27}$$

Moreover the following inequalities hold:

$$|k_2(t,s)| \le (Nk^* + N_1k_1^*)M^{-2} = k_2^*, \qquad t,s \in J, \tag{28}$$

$$|Q(t,s)| \le k_2^*(1 - 2\pi k_2^*)^{-1}, \qquad t,s \in J. \tag{29}$$

Proof. By Lemma 1, $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of linear PBVP (19) if and only if $u \in PC[J, E]$ is a solution of the impulsive integral equation

$$\begin{aligned} u(t) = {} & \int_0^{2\pi} G(t,s)(h(s) - NTu(s) - N_1Su(s))\,ds \\ & + \sum_{k=1}^{m} [G(t,t_k)(-\bar{I}_k(\eta(t_k))) + H(t,t_k)I_k(\eta(t_k))], \end{aligned} \tag{30}$$

i.e.,

$$u(t) = w(t) + \int_0^{2\pi} k_2(t,s)u(s)\,ds,$$

where $k_2(t, s)$ is given by (27),

$$w(t) = \int_0^{2\pi} G(t,s)h(s)\,ds + \sum_{k=1}^{m} [G(t,t_k)(-\bar{I}_k) + H(t,t_k)I_k]. \tag{31}$$

Inequalities (28), (29) and the series on the right hand side of (25) converging uniformly follow from paper [2]. Therefore, $Q \in C[J \times J, R]$. Let

$$Au(t) = w(t) + \int_0^{2\pi} k_2(t, s)u(s)\, ds. \tag{32}$$

Then A is an operator from $PC[J, E]$ into $PC[J, E]$. It is easy to verify that

$$\|Au - Av\|_{PC} \leq 2\pi k_2^* \|u - v\|_{PC}, \qquad u, v \in PC[J, E].$$

Since $2\pi k_2^* < 1$, A is a contractive mapping, and consequently A has a unique fixed point u in $PC[J, E]$ given by

$$u = \lim u_n \tag{33}$$

$$u_0(t) = w(t), \qquad u_n(t) = Au_{n-1}(t) \qquad (n = 1, 2, 3, \ldots). \tag{34}$$

It is not difficult to show that (33) and (34) give the same $u(t)$ as (21). ▮

LEMMA 3 [2]. *Let $k(t, s) \geq 0$ for $(t, s) \in D$ and $k_1(t, s) \geq 0$ for $(t, s) \in J \times J$. If*

$$Nk^* + N_1 k_1^* \leq M(8\pi^2 e^{\pi M})^{-1}\{[(e^{2\pi M} - 1)^2 + 16(\pi M)^2 e^{2\pi M}]^{1/2} - (e^{2\pi M} - 1)\} \tag{35}$$

is satisfied, then (20) *is satisfied and*

$$G^*(t, s) \geq 0 \qquad \text{for } (t, s) \in J \times J, \tag{36}$$

where $G^*(t, s)$ is given by (22).

COROLLARY 1 (Comparison Result). *Let $k(t, s) \geq 0$ for $(t, s) \in D$ and $k_1(t, s) \geq 0$ for $(t, s) \in J \times J$ and* (35) *be satisfied, and suppose that $u \in PC^1[J, E] \cap C^2[J', E]$ satisfies*

$$\begin{aligned} &-u'' + M^2 u \geq -NTu - N_1 Su, && t \neq t_k, \\ &\Delta u|_{t=t_k} = \theta, && \\ &\Delta u'|_{t=t_k} \leq \theta, && k = 1, 2, \ldots, m, \\ &u(0) = u(2\pi), && u'(0) = u'(2\pi). \end{aligned} \tag{37}$$

Then $u(t) \geq \theta$ for $t \in J$.

Proof. Let $h(t) = -u'' + M^2 u + NTu + N_1 Su$, $I_k = \Delta u|_{t=t_k}, \bar{I}_k =$

$\Delta u'|_{t=t_k}$. Then $h(t) \geq \theta$, $I_k = \theta$, $\bar{I}_k \leq \theta$, $k = 1, 2, \ldots, m$. By Lemmas 2 and 3, (21) and (36) hold, and therefore $u(t) \geq \theta$ for $t \in J$. ▌

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, ..., $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 2\pi]$. For $B \subset PC[J, E]$, we denote $B(t) = \{u(t) : u \in B\} \subset E$ $(t \in J)$.

LEMMA 4 [4]. *If B is bounded and the elements of B are equicontinuous on each J_k $(k = 0, 1, \ldots, m)$, then*

$$\alpha(B) = \sup_{t \in J} \alpha(B(t)),$$

where α denotes the Kuratowski measure of noncompactness.

3. EXISTENCE THEOREM

In the following, we shall use the ordered interval $[p, q]$ in space $PC[J, E]$, i.e., $[p, q] = \{u \in PC[J, E] : p \leq u \leq q$, i.e., $p(t) \leq u(t) \leq q(t)$ for all $t \in J\}$. Let us list some assumptions:

(H_1) $k(t, s) \geq 0$ for $(t, s) \in D$ and $k_1(t, s) \geq 0$ for $(t, s) \in J \times J$.

(H_2) There are $p, q \in PC^1[J, E] \cap C^2[J', E]$, $p(t) \leq q(t)$ for $t \in J$ such that

$$-p'' \leq f(t, p, Tp, Sp), t \neq t_k, \qquad p(0) = p(2\pi), \qquad p'(0) = p'(2\pi),$$
$$\Delta p|_{t=t_k} = I_k(p(t_k)), \qquad \Delta p'|_{t=t_k} \geq \bar{I}_k(p(t_k)), \qquad k = 1, 2, \ldots, m;$$
$$-q'' \geq f(t, q, Tq, Sq), t \neq t_k, \qquad q(0) = q(2\pi), \qquad q'(0) = q'(2\pi),$$
$$\Delta q|_{t=t_k} = I_k(q(t_k)), \qquad \Delta q'|_{t=t_k} \leq \bar{I}_k(q(t_k)), \qquad k = 1, 2, \ldots, m.$$

(H_3) There exist $M > 0$, $N \geq 0$, and $N_1 \geq 0$ such that

$$f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w}) \geq -M^2(u - \bar{u}) - N(v - \bar{v}) - N_1(w - \bar{w})$$

whenever $t \in J$, $p(t) \leq \bar{u} \leq u \leq q(t)$, $Tp(t) \leq \bar{v} \leq v \leq Tq(t)$, and $Sp(t) \leq \bar{w} \leq w \leq Sq(t)$.

(H_4) $I_k(u) = x_0^{(k)}$, whenever $p(t_k) \leq u \leq q(t_k)$, where $x_0^{(k)}$ is an element of E, $\bar{I}_k(u) \leq \bar{I}_k(\bar{u})$ whenever $p(t_k) \leq \bar{u} \leq u \leq q(t_k)$, $k = 1, 2, \ldots, m$.

(H_5) $Nk^* + N_1 k_1^* \leq M(8\pi^2 e^{\pi M})^{-1}\{[(e^{2\pi M} - 1)^2 + 16(\pi M)^2 e^{2\pi M}]^{1/2} - (e^{2\pi M} - 1)\}$.

THEOREM. *Let $P \subset E$ be a regular cone, and* (H_1), (H_2), (H_3), (H_4), *and*

(H_5) *be satisfied. Then there exist sequences* $\{p_n(t)\}$, $\{q_n(t)\} \subset PC^1[J, E] \cap C^2[J', E]$ *such that*

$$\begin{aligned} p(t) = p_0(t) \le p_1(t) \le \cdots \le p_n(t) \le \cdots \le q_n(t) \\ \le \cdots \le q_1(t) \le q_0(t) \le q_0(t) = q(t), \end{aligned} \tag{38}$$

and $p_n(t) \to u_*(t)$, $q_n(t) \to u^*(t)$ *as* $n \to \infty$ *uniformly in* t, u_*, $u^* \in PC^1[J, E] \cap C^2[J', E]$. *Moreover,* u_* *and* u^* *are minimal and maximal solutions of PBVP* (1) *on the ordered interval* $[p, q]$, *respectively.*

Proof. For any $\eta \in [p, q]$ (i.e., $\eta \in PC[J, E]$ and $p(t) \le \eta(t) \le q(t)$ for $t \in J$), consider linear PBVP (19), where

$$h(t) = f(t, \eta(t), T\eta(t), S\eta(t)) + M^2\eta(t) + NT\eta(t) + N_1 S\eta(t). \tag{39}$$

By Lemma 2, (19) has exactly one solution $u \in PC^1[J, E] \cap C^2[J', E]$ given by (21). Define $A\eta = u$; then A is a continuous operator from $[p, q]$ into $PC^1[J, E] \cap C^2[J', E] \subset PC[J, E]$ and η is a solution of PBVP (1) if and only if $\eta = A\eta$.

We first show that

$$p \le Ap, \qquad Aq \le q. \tag{40}$$

Let

$$p_1 = Ap \qquad \text{and} \qquad y = p_1 - p.$$

Since

$$\begin{aligned} -p_1'' + M^2(p_1 - p) &= -NT(p_1 - p) - N_1 S(p_1 - p) + f(t, p, Tp, Sp) \\ &\ge -NTy - N_1 Sy - p'', \qquad t \ne t_k, \\ p_1(0) &= p_1(2\pi), \qquad p_1'(0) = p_1'(2\pi). \end{aligned}$$

By (H_2), we obtain

$$\Delta y|_{t=t_k} = \Delta(p_1 - p)|_{t=t_k} = \theta, \qquad \Delta y'|_{t=t_k} = \Delta(p_1' - p')|_{t=t_k} \le \theta, \quad k = 1, 2, \ldots, m.$$

Therefore,

$$\begin{aligned}
&-y'' + M^2 y \geq -NTy - N_1 Sy, && t \neq t_k, \\
&\Delta y|_{t=t_k} = \theta, \qquad \Delta y'|_{t=t_k} \leq \theta, && k = 1, 2, \ldots, m, \\
&y(0) = y(2\pi), && y'(0) = y'(2\pi).
\end{aligned}$$

Hence, by Corollary 1, $y(t) \geq \theta$ for $t \in J$, i.e., $p \leq Ap$. Similarly, we can show that $Aq \leq q$ and consequently, (40) is true.

Next we prove that

$$\text{if} \qquad p \leq \eta_1 \leq \eta_2 \leq q, \qquad \text{then} \qquad A\eta_1 \leq A\eta_2. \tag{41}$$

Let $y_i = A\eta_i$ and $z = y_2 - y_1$. We have

$$\begin{aligned}
&-y_i'' + M^2 y_i = -NTy_i - N_1 Sy_i + h_i(t), && t \neq t_k, \\
&\Delta y_i|_{t=t_k} = I_k(\eta_i(t_k)), && \\
&\Delta y_i'|_{t=t_k} = \bar{I}_k(\eta_i(t_k)), && k = 1, 2, \ldots, m. \\
&y_i(0) = y_i(2\pi), \qquad y_i'(0) = y_i'(2\pi) && (i = 1, 2),
\end{aligned}$$

where

$$\begin{aligned}
h_i(t) = f(t, \eta_i(t), T\eta_i(t), S\eta_i(t)) + M^2 \eta_i(t) \\
+ NT\eta_i(t) + N_1 S\eta_i(t), \qquad i = 1, 2.
\end{aligned}$$

By virtue of (H_3) and (H_4), we see that $h_1(t) \leq h_2(t)$ for $t \in J$, and therefore,

$$\begin{aligned}
&-z'' + M^2 z \geq -NTz - N_1 Sz, && t \neq t_k, \\
&\Delta z|_{t=t_k} = \theta, \qquad \Delta z'|_{t=t_k} \leq \theta, && k = 1, 2, \ldots, m, \\
&z(0) = z(2\pi), && z'(0) = z'(2\pi),
\end{aligned}$$

which implies by Corollary 1 that $z(t) \geq \theta$ for $t \in J$, i.e., (41) holds.

Now, let $p_0 = p$, $q_0 = q$, $p_n = Ap_{n-1}$, $q_n = Aq_{n-1}$ $(n = 1, 2, \ldots)$. It follows from (40) and (41) that (38) holds. By definition of A, we have

$$\begin{aligned}
p_n(t) = \int_0^{2\pi} G^*(t, s) h_{n-1}(s)\, ds + \sum_{k=1}^{m} [G^*(t, t_k)(-\bar{I}_k(p_{n-1}(t_k))) \\
+ H^*(t, t_k) I_k(p_{n-1}(t_k))],
\end{aligned}$$

where

$$h_{n-1}(t) = f(t, p_{n-1}(t), Tp_{n-1}(t), Sp_{n-1}(t)) + M^2 p_{n-1}(t) + NTp_{n-1}(t) + N_1 Sp_{n-1}(t).$$

Finally, we shall show that $p_n(t) \to u_*(t)$, $q_n(t) \to u^*(t)$ as $n \to \infty$ uniformly in t, u_*, $u^* \in PC^1[J, E] \cap C^2[J', E]$ are minimal and maximal solutions of PBVP (1) on the ordered interval $[p, q]$, respectively.

Let $B = \{p_n\} \subset [p, q]$, $B(t) = \{p_n(t)\} \subset E$, $t \in J$. In the following, we shall show that B is relatively compact. By virtue of the regularity of P and (38), we first show that $B(t)$ is relatively compact, i.e.,

$$\alpha(B(t)) = 0, \qquad t \in J, \tag{42}$$

and then, we obtain that K is a normal cone in $PC[J, E]$. Therefore, $[p, q]$ is a bounded set in $PC[J, E]$. For any $\eta \in [p, q]$, by (H_2), (H_3), and (H_4), we have

$$\begin{aligned} -p'' + M^2 p + NTp + N_1 Sp &\le f(t, p, Tp, Sp) + M^2 p + NTp + N_1 Sp \\ &\le f(t, \eta, T\eta, S\eta) + M^2 \eta + NT\eta + N_1 S\eta \\ &\le f(t, q, Tq, Sq) + M^2 q + NTq + N_1 Sq \\ &\le -q'' + M^2 q + NTq + N_1 Sq, \\ \Delta q'|_{t=t_k} &\le \bar{I}_k(q(t_k)) \le \bar{I}_k(\eta(t_k)) \\ &\le \bar{I}_k(p(t_k)) \le \Delta p'|_{t=t_k}. \end{aligned}$$

By the normality of K, $\{f(t, \eta, T\eta, S\eta) + M^2\eta + NT\eta + N_1 S\eta \mid \eta \in [p, q]\}$ and $\{\bar{I}_k(\eta(t_k)) \mid \eta \in [p, q]\}$ are bounded sets in $PC[J, E]$ and in E, respectively. On the other hand, $\{p_n(t) \mid n \in N\}$ satisfies

$$\begin{aligned} p_n'(t) = &-\int_0^{2\pi} H(t, s)[-NTp_n(s) - N_1 Sp_n(s) + h_{n-1}(s)]\, ds \\ &+ \sum_{k=1}^{m} [H(t, t_k)\bar{I}_k(p_{n-1}(t_k) - M^2 G(t, t_k) I_k(p_{n-1}(t_k))], \end{aligned} \tag{43}$$

where

$$h_{n-1}(t) = f(t, p_{n-1}(t), Tp_{n-1}(t), Sp_{n-1}(t)) + M^2 p_{n-1}(t) + NTp_{n-1}(t) + N_1 Sp_{n-1}(t).$$

Noting the fact that

$$\max_{(t,s)\in J\times J} |H(t,s)| \le (1 + e^{2\pi M})[2(e^{2\pi M} - 1)]^{-1},$$

$$\max_{(t,s)\in J\times J} G(t,s) = (1 + e^{2\pi M})[2M(e^{2\pi M} - 1)]^{-1},$$

we have $\{p_n'(t) \mid n \in N\}$ is a bounded set in $PC[J, E]$. Applying the mean value theorem, we obtain that the elements of $B = \{p_n\}$ are equicontinuous on each J_k, $k = 1, 2, \ldots, m$. By Lemma 4 and (42), we have

$$\alpha(B) = \sup_{t\in J} \alpha(B(t)) = 0.$$

Hence, $\{p_n\}$ is a relatively compact set in $PC[J, E]$. In view of the normality of K and (38), $\{p_n\}$ in $PC[J, E]$ converges to $u_* \in [p, q]$, i.e., $\{p_n(t)\}$ converges to $u_*(t)$ uniformly on J. Since A is a continuous operator, we get that u_* is a solution of PBVP (1).

Similarly, we can show that $\{q_n\}$ in $PC[J, E]$ converges to $u^* \in PC[J, E]$ and that u^* is a solution of PBVP (1). It follows by using standard arguments in paper [3] that u_*, u^* are minimal and maximal solutions of PBVP (1) on the ordered interval $[p, q]$, respectively. ∎

Remark 1. If $E = R$, $I_k = \bar{I}_k = \theta$, $k = 1, 2, \ldots, m$, then this theorem is the main result in paper [2].

Remark 2. By Theorem 2.2 in paper [5], if E is weakly complete and P is normal, then P is regular. Hence, the main result in our paper for the case in which E is weakly complete and P is normal still holds.

4. An Example

Consider the following PBVP of mixed type in real space R.

$$\begin{aligned} -u'' &= \tfrac{1}{10}(1 - u)^5 - 12\pi e^{-12\pi^2}\left(\int_0^t e^{ts}u(s)\,ds\right)^2 + 4e^{-4\pi}\int_0^{2\pi} |\sin(s)|u(s)\,ds \\ &\quad - \pi^{-2}e^{-4\pi}\left(\int_0^{2\pi} |\sin(s)|u(s)\,ds\right)^3, \qquad t \ne t_k, \\ &\Delta u|_{t=\pi/2} = \tfrac{1}{2}, \qquad \Delta u|_{t=3\pi/2} = -\tfrac{1}{2}, \\ &\Delta u'|_{t=\pi/2} = \Delta u'|_{t=3\pi/2} = 0, \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{aligned} \tag{44}$$

We show that PBVP (44) has minimal maximal solutions satisfying $p(t) \leq u(t) \leq q(t)$ for $0 \leq t \leq 2\pi$, where

$$p(t) = \begin{cases} 0, 0 \leq t \leq \pi/2 \\ \frac{1}{2}, \pi/2 < t \leq 3\pi/2 \\ 0, 3\pi/2 < t \leq 2\pi, \end{cases} \qquad q(t) = \begin{cases} \frac{3}{2}, 0 \leq t \leq \pi/2 \\ 2, \pi/2 < t \leq 3\pi/2 \\ \frac{3}{2}, 3\pi/2 < t \leq 2\pi. \end{cases}$$

In fact, this is a PBVP of type (1), where

$$f(t, u, v, w) = \tfrac{1}{10}(1 - u)^5 - 12\pi e^{-12\pi^2} v^2 + 4e^{-4\pi} w - \pi^{-2} e^{-4\pi} w^3$$

and

$$Tu(t) = \int_0^t e^{ts} u(s)\, ds, \qquad Su(t) = \int_0^{2\pi} |\sin(s)| u(s)\, ds.$$

Obviously, (H_1) and (H_4) are satisfied.

$Tp \leq e^{4\pi^2}(4\pi)^{-1}$, $Sp = 1$, $Sq = 7$. Therefore, $f(t, p, Tp, Sp) \geq 0$, $f(t, q, Tq, Sq) \leq 0$. Hence, $-p'' \leq f(t, p, Tp, Sp)$, $-q'' \geq f(t, q, Tq, Sq)$, $t \neq t_k$, $k = 1, 2, \ldots, m$. It is easy to see that the other conditions in (H_2) are satisfied.

In addition,

$$\begin{aligned} f'_u &= -1/2(1-u)^4 \geq -\tfrac{1}{2} \geq -16, \\ &\quad \text{whenever } 0 \leq t \leq 2\pi \text{ and } p(t) \leq u \leq q(t), \\ f'_v &= -24\pi e^{-12\pi^2} v \geq -24 e^{-8\pi^2}, \\ &\quad \text{whenever } 0 \leq v \leq Tq \leq e^{4\pi^2}/\pi, \\ f'_w &= 4e^{-4\pi} - 3\pi^{-2} e^{-4\pi} w^2 \geq -13 e^{-4\pi}, \\ &\quad \text{whenever } 0 \leq w \leq Sq = 7. \end{aligned}$$

Consequently, (H_3) is satisfied for $M = 4$, $N = 24e^{-8\pi^2}$, and $N_1 = 13e^{-4\pi}$. Finally, we verify that (H_5) holds. Denote the left and right sides of (H_5) by l and r, respectively. Since $k(t, s) = e^{ts}$ and $k_1(t, s) = |\sin(s)|$, we have $k^* = e^{4\pi^2}$, $k_1^* = 1$, and so

$$l = 24e^{-8\pi^2 + 4\pi^2} + 13e^{-4\pi} < 24e^{-4\pi} + 13e^{-4\pi} = 37e^{-4\pi}.$$

By paper [2], we know that

$$\begin{aligned} r &= 4(8\pi^2 e^{4\pi})^{-1}\{[(e^{8\pi} - 1)^2 + (16\pi)^2 e^{8\pi}]^{1/2} - (e^{8\pi} - 1)\} \\ &> 64(e^{4\pi} + 8\pi)^{-1} > 64(e^{4\pi} + e^{4\pi}/2)^{-1} > 42e^{-4\pi}. \end{aligned}$$

Hence $r > l$, i.e., (H_5) is satisfied and therefore our conclusion follows from the existence theorem.

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